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Analogues of Fekete and Descartes Systems of Solutions for Difference Equations

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1. INTRODUCTION

In this paper, either $I=\{a, a+1, ..., b\}$, if $b-a$ is a positive integer, or $I=\{a, a+1, ...\}$. Given $n \geqslant 1$, for $0 \leqslant j \leqslant n$, let $I^j=\{a, a+1, ..., b+j\}$ in the former definition of I, and let $I^j=I$ in the latter case. For a finite or infinite sequence u: $u(a)$, $u(a+1), ...,$ defined on some I^m, Hartman [7] defined $s=a$ to be a *generalized zero* of u if $u(a)=0$, and $s>a$ to be a *generalized zero* of u if either $u(s)=0$ or there exists an integer j, $1 \leqslant j \leqslant s-a$, such that $(-1)^j u(s-j)\, u(s)>0$, and if $j>1$, $u(s-j+1)= \cdots =u(s-1)=0$. We shall be concerned with characterizing solutions, in terms of generalized zeros of higher order differences, for the nth order linear difference equation

$$Pu(s)=\sum_{j=0}^{n} \alpha_j(s)\, u(s+j)=0, \tag{1}$$

where s ranges over I, $\alpha_n(s) \equiv 1$, $\alpha_0(s) \neq 0$ on I, and the coefficients $\alpha_j(s)$, $0 \leqslant j \leqslant n$, are defined on I. A solution u of (1) is then defined on I^n.

In his landmark paper, Hartman [7] defined the difference equation (1) to be *disconjugate on I^n* if and only if the only solution of (1) having n generalized zeros on I^n is the trivial solution. In determining criteria for the disconjugacy of (1) on I^n, Hartman established several conditions

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analogous to those for the disconjugacy of a linear nth order ordinary differential equation. Among those, he obtained a Pólya [10] criterion (or Markov condition [2]), and other criteria concerning the positivity of minors of a Wronskian determinant for the disconjugacy of (1) on I^n. He also obtained a criterion concerning the unique solvability of a class of boundary value problems for the disconjugacy of (1) on I^n.

Muldowney [8] and Eloe and Henderson [4] studied criteria for the right disfocality of an nth order linear ordinary differential equation; in doing so, they obtained several necessary and sufficient conditions in terms of the positivity of minors of a Wronskian determinant, many of which are analogues of sign conditions associated with Markov, Descartes, and Fekete conditions [1, 2].

For a sequence u defined on I^n, define differences by $\Delta u(s) = u(s+1) - u(s)$ on I^{n-1}, and for $2 \leqslant i \leqslant n$, $\Delta^i u(s) = \Delta(\Delta^{i-1} u(s))$ on I^{n-i}. Motivated by the results for linear ordinary differential equations in [4, 8], Eloe [3] defined the linear difference equation (1) to be *right disfocal on* I^n if and only if $u \equiv 0$ is the only solution of (1) on I^n such that $\Delta^{j-1} u$ has a generalized zero at s_j, $1 \leqslant j \leqslant n$, where $a \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$ in I^1. Eloe [3] then formulated criteria for the right disfocality of (1) on I^n in analogy to the Markov, Descartes, and Fekete conditions given in [4, 8].

As further motivation for this paper, we cite the extensive paper of Muldowney [9], in which he applied these types of positivity criteria to a large class of boundary value problems for nth order linear differential equations. This large class of problems was formulated in terms of right $(m_1; \ldots; m_l)$ invertibility and included both the conjugate and right focal types of boundary value problems.

Our study will be concerned with criteria for what we shall call $m_1, \ldots, m_l$ right disfocaility of (1) on I^n.

DEFINITION 1.1. (a) Let $1 \leqslant l \leqslant n$ and $m_1, \ldots, m_l$ be positive integers such that $\sum_{i=1}^{l} m_i = n$. We say that (1) is $m_1, \ldots, m_l$ *right disfocal on* I^n if and only if $u \equiv 0$ is the only solution of (1) on I^n such that, for each $1 \leqslant i \leqslant l$, $\Delta^{i-1} u$ has m_i generalized zeros at

$$s_{m_1 + \cdots + m_{i-1} + 1}, \ldots, s_{m_1 + \cdots + m_i},$$

where

$$a \leqslant s_1 < \cdots < s_{m_1} \text{ in } I^{n-l+1}, \text{ and}$$

$$s_{m_1 + \cdots + m_{i-1}} \leqslant s_{m_1 + \cdots + m_{i-1} + 1} < \cdots < s_{m_1 + \cdots + m_i} \qquad (2)$$

$$\text{in } I^{n-l+1}, 2 \leqslant i \leqslant l.$$

(b) If, for some $m_1, \ldots, m_l$, (1) is not $m_1, \ldots, m_l$ right disfocal on I^n

and if u is a nontrivial solution of (1) on I^n such that $\Delta^{i-1}u$ has m_i generalized zeros at $s_{m_1+\cdots+m_{i-1}+1}, ..., s_{m_1+\cdots+m_i}$, $1 \leqslant i \leqslant l$, where $\{s_j\}_{j=1}^n$ satisfies (2), then we shall call u an $m_1, ..., m_l$ *right focal solution of* (1) *on* I^n having an $m_1, ..., m_l$ right distribution of generalized zeros at $\{s_j\}_{j=1}^n$. If $\{s_j\}_{j=1}^n \subseteq X$, where X is some set, we shall say that u has an $m_1, ..., m_l$ right distribution of generalized zeros on X.

Hartman [7, Proposition 5.1] obtained a discrete version of Rolle's theorem with respect to generalized zeros. Thus, it follows that if (1) is right disfocal on I^n, then (1) is $m_1, ..., m_l$ right disfocal on I^n, for all $m_1, ..., m_l$. In turn, if (1) is $m_1, ..., m_l$ right disfocal, for some $m_1, ..., m_l$, then (1) is disconjugate on I^n.

The object of this paper is to obtain criteria for the $m_1, ..., m_l$ right disfocality of (1) on I^n in terms of positivity conditions on minors of Wronskian determinants. These criteria are analogues to those criteria for the disconjugacy and right disfocality of (1) on I^n given by Hartman [7] and Eloe [3], respectively. In Section 2, we shall introduce further notation and establish some general positivity conditions on minors of determinants of interest. Then, in Section 3, we shall establish our criteria for the $m_1, ..., m_l$ right disfocality of (1) on I^n.

2. Notation and Preliminary Lemmas

In this section, Eq. (1) is not involved. We introduce notation and establish some positivity conditions on minors of certain determinants via the use of a standard identity on determinants.

Let $A = [a_{ij}]_{1 \leqslant i,j \leqslant n}$ be a real $n \times n$ matrix. For $1 \leqslant k \leqslant n$ and indices $1 \leqslant i_1 < \cdots < i_k \leqslant n$, define

$$D^k(i_1, ..., i_k) \equiv \det[a_{i_j,l}]_{1 \leqslant j,l \leqslant k},$$

and for $b = (b_1, ..., b_k) \in \mathbf{R}^k$ and $1 \leqslant j \leqslant k$, let

$$D_j^k(i_1, ..., b, ..., i_{k-1})$$

denote the determinant of the $k \times k$ matrix where the lth row is

$$(a_{i_l,1}, ..., a_{i_l,k}), \qquad 1 \leqslant l \leqslant j-1,$$

the jth row is

$$(b_1, ..., b_k),$$

and the lth row is

$$(a_{i_{l-1},1}, ..., a_{i_{l-1},k}), \qquad j+1 \leqslant l \leqslant k.$$

The proof of the following lemma is an application of Sylvester's identity [5]; see [4, 8] for a proof of the lemma.

LEMMA 2.1. *Let $A = [a_{ij}]$ be an $n \times n$ real matrix. Let $2 \leqslant k \leqslant n$, indices $1 \leqslant i_1 < \cdots < i_k \leqslant n$, and $b \in \mathbf{R}^k$ be given. Then, for each $2 \leqslant j \leqslant k$,*

$$\begin{aligned} &D_{j-1}^{k-1}(i_2, ..., b, ..., i_{k-1})\, D^k(i_1, ..., i_k) \\ &\quad = D^{k-1}(i_1, ..., i_{k-1})\, D_{j-1}^k(i_2, ..., b, ..., i_k) \\ &\quad + D^{k-1}(i_2, ..., i_k)\, D_j^k(i_1, ..., b, ..., i_{k-1}). \end{aligned}$$

Lemma 2.1 plays a fundamental role in establishing the next lemma which in turn will be used in establishing positivity conditions on determinants involving systems of solutions of (1). Let $r_1, ..., r_n$ be positive integers such that

$$n \geqslant r_1 \geqslant \cdots \geqslant r_k \geqslant r_{k+1} \geqslant \cdots \geqslant r_n = 1 \text{ and } r_k \leqslant r_{k+1} + 1,\ 1 \leqslant k \leqslant n-1. \tag{3}$$

We point out here that if $r_k = n - k + 1$, $1 \leqslant k \leqslant n$, then the following lemma is equivalent to a lemma established by Eloe [3, Lemma 2.2].

LEMMA 2.2. *Let $A = [a_{ij}]$ be given and let $\{r_k\}_{k=1}^n$ satisfy* (3). *Then*

$$D^k(i, ..., i+k-1) > 0, \qquad 1 \leqslant i \leqslant r_k,\ 1 \leqslant k \leqslant n, \tag{4}$$

if and only if

$$\begin{aligned} &D^{h+k}(i_1, ..., i_h, i, ..., i+k-1) > 0, \\ &\qquad 1 \leqslant i_1 < \cdots < i_h < i \leqslant r_k,\ 0 \leqslant h,\ 1 \leqslant k \leqslant n. \end{aligned} \tag{5}$$

Proof. By $h = 0$, we mean $D^{h+k}(i_1, ..., i_h, i, ..., i+k-1) = D^k(i, ..., i+k-1)$, so it is clear that (5) implies (4). The argument now proceeds by an induction on k, h, and the difference $i - i_1$.

First, let $k = 1$. If $h = 0$, there is nothing to prove. So assume that $0 < h$ and that, for all $0 \leqslant l < h$,

$$D^{l+1}(i_1, ..., i_l, j) > 0, \quad 1 \leqslant i_1 < \cdots < i_l < j \leqslant r_1.$$

Moreover, if $i - i_1 = h$, then

$$D^{h+1}(i_1, ..., i_h, i) = D^{h+1}(i-h, ..., i-1, i) > 0$$

by (4), since $1 \leqslant i-h \leqslant r_{h+1}$. (To see that $i-h \leqslant r_{h+1}$, note that it follows from (3) $r_1 \leqslant r_{h+1}+h$. Thus $i-h \leqslant r_1 - h \leqslant r_{h+1}$.) Thus, let $\alpha > h$ and assume, in addition to the hypothesis on h, that $D^{h+1}(i_1, ..., i_h, j) > 0$, for all sets of indices satisfying $1 \leqslant i_1 < \cdots < i_h < j \leqslant r_1$, where $h \leqslant j - i_1 < \alpha$.

Now suppose $1 \leqslant i_1 < \cdots < i_h < i \leqslant r_1$ and that $i - i_1 = \alpha$. Since $i - i_1 > h$, there are two cases to consider.

Case (i). $i_h + 1 = i$. Then, for some $2 \leqslant j \leqslant h$, $i_j - i_{j-1} > 1$. Apply Lemma 2.1 with $b = (a_{i_{j-1}+1,1}, ..., a_{i_{j-1}+1,h+1})$ and let $i_h = i-1$ when appropriate. Then

$$\begin{aligned} &D_{j-1}^{h-1+1}(i_2, ..., b, ..., i_{h-1}, i-1)\, D^{h+1}(i_1, ..., i_h, i) \\ &\quad = D^{h-1+1}(i_1, ..., i_{h-1}, i-1) \\ &\qquad \times D_{j-1}^{h+1}(i_2, ..., b, ..., i_{h-1}, i-1, i) \\ &\qquad + D^{h-1+1}(i_2, ..., i_{h-1}, i_h, i) \\ &\qquad \times D_j^{h+1}(i_1, ..., b, ..., i_{h-1}, i-1). \end{aligned}$$

The factor D_{j-1}^{h-1+1} and each of the factors involving D^{h-1+1} are positive by induction on h. Also, since $i - i_2 < \alpha$ and $i - 1 - i_1 < \alpha$, the factors D_{j-1}^{h+1} and D_j^{h+1} are positive. Consequently, $D^{h+1}(i_1, ..., i_h, i) > 0$ for this case.

Case (ii). $i - i_h > 1$. This time set $b = (a_{i-1,1}, ..., a_{i-1,h+1})$ and apply Lemma 2.1 (and writing $i-1$ rather than b). Then

$$\begin{aligned} &D_h^{h-1+1}(i_2, ..., i_h, i-1)\, D^{h+1}(i_1, ..., i_h, i) \\ &\quad = D^{h-1+1}(i_1, ..., i_{h-1}, i_h)\, D_h^{h+1}(i_2, ..., i_h, i-1, i) \\ &\qquad + D^{h-1+1}(i_2, ..., i_h, i)\, D_{h+1}^{h+1}(i_1, ..., i_h, i-1). \end{aligned}$$

Here, the factor D_h^{h-1+1} and each of the factors labeled D^{h-1+1} are positive by induction on h, and since $i - i_2 < \alpha$ and $i - 1 - i_1 < \alpha$, the factors D_h^{h+1} and D_{h+1}^{h+1} are also positive. Again, we conclude $D^{h+1}(i_1, ..., i_h, i) > 0$.

Inducting now on k, assume $1 < k \leqslant n$ and that, for $1 \leqslant s < k$,

$$D^{t+s}(i_1, ..., i_t, j, ..., j+s-1) > 0,$$
$$1 \leqslant i_1 < \cdots < i_t < j \leqslant r_s, 0 \leqslant t.$$

Inducting again on h, our arguments proceed much like those above in Cases (i) and (ii). If $h = 0$, again there is nothing to prove. So assume $0 < h$ and that, for all $0 \leqslant l < h$,

$$D^{l+k}(i_1, ..., i_l, j, ..., j+k-1) > 0, \quad 1 \leqslant i_1 < \cdots < i_l < j \leqslant r_k.$$

Moreover, if i and i_1 are indices such that $i-i_1=h$, and since $1 \leqslant i-h \leqslant r_{h+k}$, then from condition (4),

$$D^{h+k}(i_1, ..., i_h, i, ..., i+k-1)$$
$$= D^{h+k}(i-h, ..., i-1, i, ..., i+k-1) > 0.$$

Thus, let $\alpha > h$ and assume, in addition to the assumptions on k and h, that $D^{h+k}(i_1, ..., i_h, j, ..., j+k-1) > 0$, for all sets of indices satisfying $1 \leqslant i_1 < \cdots < i_h < j \leqslant r_k$, where $h \leqslant j-i_1 < \alpha$.

Now suppose $1 \leqslant i_1 < \cdots < i_h < i \leqslant r_k$ and that $i-i_1=\alpha$. Since $i-i_1>h$, the same cases as above arise.

Case (iii). $i_h+1=i$. Then, for some $2 \leqslant j \leqslant h$, $i_j - i_{j-1} > 1$. Setting $b=(a_{i_{j-1}+1,1}, ..., a_{i_{j-1}+1,h+k})$, using $i_h=i-1$, and applying Lemma 2.1, we have

$$D_{j-1}^{h-1+k}(i_2, ..., b, ..., i_{h-1}, i-1, i, ..., i+k-2)$$
$$\times D^{h+k}(i_1, ..., i_h, i, ..., i+k-1)$$
$$= D^{h-1+k}(i_1, ..., i_{h-1}, i-1, i, ..., i+k-2)$$
$$\times D_{j-1}^{h+k}(i_2, ..., b, ..., i_{h-1}, i-1, i, ..., i+k-1)$$
$$+ D^{h-1+k}(i_2, ..., i_{h-1}, i_h, i, ..., i+k-1)$$
$$\times D_j^{h+k}(i_1, ..., b, ..., i_{h-1}, i-1, i, ..., i+k-2).$$

The factor D_{j-1}^{h-1+k} and each of the factors involving D^{h-1+k} are positive by the inductive assumption on h. Furthermore, since $i-i_2<\alpha$ and $i-1-i_1<\alpha$, the factors D_{j-1}^{h+k} and D_j^{h+k} are positive. Consequently, $D^{h+k}(i_1, ..., i_h, i, ..., i+k-1) > 0$ for this case.

Case (iv). $i-i_h>1$. This time we set $b=(a_{i-1,1}, ..., a_{i-1,h+k})$. Then applying Lemma 2.1 (and again writing $i-1$ rather than b), we have $D_h^{h-1+k}(i_2, ..., i_h, i-1, i, ..., i+k-2)\, D^{h+k}(i_1, ..., i_h, i, ..., i+k-1) = D^{h+k-1}(i_1, ..., i_h, i, ..., i+k-2)\, D_n^{h+k}(i_2, ..., i_h, i-1, i, ..., i+k-1) + D^{h-1+k}(i_2, ..., i_h, i, ..., i+k-1)\, D_{h+1}^{h+k}(i_1, ..., i_h, i-1, i, ..., i+k-2)$. In this situation, the factors D_h^{h-1+k} and D^{h-1+k} are positive by induction on h, the factor D^{h+k-1} is positive by induction on k, and since $i-i_2<\alpha$ and $i-1-i_1<\alpha$, the factors D_h^{h+k} and D_{h+1}^{h+k} are also positive. Consequently, we again conclude $D^{h+k}(i_1, ..., i_h, i, ..., i+k-1) > 0$. In conclusion, condition (5) is satisfied and the proof is complete.

Let $u_1, ..., u_n$ be sequences defined on I^n. For $1 \leqslant k \leqslant n$ and indices $1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n$, define

$$D^k(i_1, ..., i_k)(s) \equiv \det[\Delta^{i_j-1} u_l(s)]_{1 \leqslant j, l \leqslant k},$$

where $s \in I^{n-i_k+1}$, and define

$$D^k(i_1, ..., i_k; s_1, ..., s_k) \equiv \det[\Delta^{i_j-1} u_l(s_j)]_{1 \leqslant j,l \leqslant k},$$

where $a \leqslant s_1 \leqslant \cdots \leqslant s_k$ in I^{n-i_k+1}.

Lemma 2.3 follows immediately from Lemma 2.2.

LEMMA 2.3. *Let $u_1, ..., u_n$ be sequences defined on I^n, and let $\{r_k\}_{k=1}^n$ satisfy* (3). *Then*

$$D^k(i, ..., i+k-1)(s) > 0, \quad s \in I^{n-i-k+2}, 1 \leqslant i \leqslant r_k, 1 \leqslant k \leqslant n, \tag{6}$$

if and only if

$$D^{h+k}(i_1, ..., i_h, i, ..., i+k-1)(s) > 0,$$
$$s \in I^{n-i-k+2}, 1 \leqslant i_1 < \cdots < i_h < i \leqslant r_k, 0 \leqslant h, 1 \leqslant k \leqslant n. \tag{7}$$

Remark. To be consistent with terminology employed in [4], we shall say that a system of sequences, $u_1, ..., u_n$, defined on I^n, is a *F*-system (for Fekete) with respect to $\{r_k\}$ if (6) holds and that the system is a D-system (for Descartes) with respect to $\{r_k\}$ if (7) holds.

3. CRITERIA FOR $m_1, ..., m_l$ RIGHT DISFOCALITY

In this section, we formulate necessary and sufficient conditions for the $m_1, ..., m_l$ right disfocality of (1) on I^n. Before this formulation, we shall provide three principal tools, the first of which is a discrete version of Rolle's Theorem; see Hartman [7, Proposition 5.1].

PROPOSITION 3.1. *Suppose that the finite sequence $u(1), ..., u(j)$ has N_j generalized zeros and that the finite sequence $\Delta u(1), ..., \Delta u(j-1)$ has M_j generalized zeros. Then $M_j \geqslant N_j - 1$.*

PROPOSITION 3.2. *Let γ be a positive integer. Let u be a sequence defined on I^n and suppose that u has γ generalized zeros at $(a \leqslant)\ s_1 < \cdots < s_\gamma$ in I^n. Then, for any partition by positive integers $(m_1, ..., m_i)$ of γ (i.e., $\sum_{j=1}^i m_j = \gamma$), u has an $m_1, ..., m_i$ right distribution of generalized zeros on $\{s_1, ..., s_\gamma - i + 1\}$.*

Proof. Assume that u has γ generalized zeros at $(a \leqslant)\ s_1 < \cdots < s_\gamma$ and that $m_1, ..., m_i$ are positive integers such that $\sum_{j=1}^i m_j = \gamma$. Then u has m_1 generalized zeros at $s_1, ..., s_{m_1}$ and $\gamma - m_1 + 1$ generalized zeros on $\{s_{m_1}, ..., s_\gamma\}$. By Proposition 3.1, Δu has at least $\gamma - m_1$ generalized zeros at

$(s_{m_1} \leqslant)\ t_{m_1+1} < \cdots < t_\gamma \leqslant s_\gamma - 1$. Thus, Δu has m_2 generalized zeros at $t_{m_1+1}, ..., t_{m_1+m_2}$ and at least $\gamma - m_1 - m_2 + 1$ generalized zeros on $\{t_{m_1+m_2}, ..., s_\gamma - 1\}$.

Continuing this argument, it can be shown that, for each $2 \leqslant j < i$, $\Delta^{j-1}u$ has m_j generalized zeros at $(\hat{s}_{m_1+\cdots+m_{j-1}} \leqslant)\ \hat{s}_{m_1+\cdots+m_{j-1}+1} < \cdots < \hat{s}_{m_1+\cdots+m_j}$ and at least $\gamma - m_1 - \cdots - m_j + 1$ generalized zeros on $\{\hat{s}_{m_1+\cdots+m_j}, ..., s_\gamma - j + 1\}$. Thus, assume $\Delta^{i-2}u$ has m_{i-1} generalized zeros at $(\hat{s}_{m_1+\cdots+m_{i-2}} \leqslant)\ \hat{s}_{m_1+\cdots+m_{i-2}+1} < \cdots < \hat{s}_{m_1+\cdots+m_{i-1}}$ and at least $\gamma - m_1 - \cdots - m_{i-1} + 1$ generalized zeros on $\{\hat{s}_{m_1+\cdots+m_{i-1}}, ..., s_\gamma - (i-1) + 1\}$. Apply Proposition 3.1 and $\Delta^{i-1}u$ has at least $\gamma - m_1 - \cdots - m_{i-1} = m_i$ generalized zeros on $\{\hat{s}_{m_1+\cdots+m_{i-1}}, ..., s_\gamma - i + 1\}$. In summary, then, u has an $m_1, ..., m_i$ right distribution of generalized zeros on $\{s_1, ..., s_\gamma - i + 1\}$.

For the remainder of this paper, let $2 \leqslant l \leqslant n$, and let $m_1, ..., m_l$ be positive integers such that $\sum_{i=1}^{l} m_i = n$. For each $1 \leqslant k \leqslant n$, define

$$r_k = \begin{cases} l, & \text{if } 1 \leqslant k \leqslant m_l, \\ l-j, & \text{if } m_l + \cdots + m_{l-j+1} + 1 \leqslant k \leqslant m_l + \cdots + m_{l-j},\ 1 \leqslant j \leqslant l-1. \end{cases}$$

Note that $\{r_k\}$ satisfies (3).

The proof of the next proposition is similar to, but much more tedious than, the proof of a result given in Eloe [3, Proposition 3.2]. Thus, we state the next proposition without proof.

PROPOSITION 3.3. *Let u be a sequence defined on I^n such that u has an $m_1, ..., m_l$ right distribution of generalized zeros at $\{s_1, ..., s_n\}$, where $\{s_j\}_{j=1}^n$ satisfies* (2). *Then, for each $1 \leqslant k \leqslant n$, there exists $\{\sigma_1, ..., \sigma_k\} \subseteq I^n$ such that, if $1 \leqslant k \leqslant m_1$, then $a \leqslant \sigma_1 < \cdots < \sigma_k$ in I^n and*

$$(-1)^{k-j+1}\, u(\sigma_j) \geqslant 0, \qquad 1 \leqslant j \leqslant k,$$

and if $m_1 + \cdots + m_{\alpha-1} + 1 \leqslant k \leqslant m_1 + \cdots + m_\alpha$, for some $2 \leqslant \alpha \leqslant l$, then

$$a \leqslant \sigma_1 < \cdots < \sigma_{m_1},$$

$$(\sigma_{m_1+\cdots+m_{\beta-1}} \leqslant)\ \sigma_{m_1+\cdots+m_{\beta-1}+1} < \cdots < \sigma_{m_1+\cdots+m_\beta}, \qquad 2 \leqslant \beta < \alpha, \tag{8}$$

$$(\sigma_{m_1+\cdots+m_{\alpha-1}} \leqslant)\ \sigma_{m_1+\cdots+m_{\alpha-1}+1} < \cdots < \sigma_k \quad \text{in} \quad I^{n-\alpha+1},$$

and

$$(-1)^{k-j+1}\, \Delta^{i-1}u(\sigma_j) \geqslant 0, \tag{9}$$

for each pair of indices $1 \leqslant i \leqslant \alpha$ and $1 \leqslant j \leqslant k$ satisfying $m_1 + \cdots + m_{i-1} + 1 \leqslant j \leqslant m_1 + \cdots + m_i$.

Remark. As is shown in Eloe's proof [3, Proposition 3.2], it can be shown above that $\sigma_j = s_j$ or $\sigma_j = s_j - 1$, $1 \leqslant j \leqslant k$.

We now present the main result of this paper.

THEOREM 3.4. *The following are equivalent*:

(i) (1) *is* $m_1, ..., m_l$ *right disfocal on* I^n;

(ii) (1) *has an* F-*system with respect to* $\{r_k\}$ *of solutions on* I^n;

(iii) (1) *has a* D-*system with respect to* $\{r_k\}$ *of solutions on* I^n;

(iv) *there exists a system of solutions* $u_1, ..., u_n$ *of* (1) *on* I^n *such that*

$$D^k(i_1, ..., i_k; s_1, ..., s_k) > 0$$

for all sets of indices satisfying $1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant l$, $i_j \leqslant r_{k-j+1}$, $1 \leqslant j \leqslant k$, *and for all points* $\{s_j\}_{j=1}^k$ *satisfying* $a \leqslant s_j < s_{j+1}$ *in* $I^{n-i_{j+1}+1}$, *if* $i_j = i_{j+1}$, *and* $a \leqslant s_j \leqslant s_{j+1}$ *in* $I^{n-i_{j+1}+1}$, *if* $i_j < i_{j+1}$, $1 \leqslant j \leqslant k-1$, $1 \leqslant k \leqslant n$.

Proof. The pattern of the proof is to show that (i) implies (ii), that (ii) is equivalent to (iii), that (ii) is equivalent to (iv), and that (iv) implies (i).

For (i) implies (ii), assume that (1) is $m_1, ..., m_l$ right disfocal on I^n. Let $u_1, ..., u_n$ be a system of solutions of (1) on I^n satsifying the partial set of initial conditions

$$\begin{aligned} \Delta^{i-1}u_k(a) = 0, \qquad & 1 \leqslant i \leqslant n-k, \\ (-1)^{k-1}\Delta^{n-k}u_k(a) > 0, \qquad & 1 \leqslant k \leqslant n. \end{aligned} \tag{10}$$

Note that $D^k(i, ..., i+k-1)(s) = 0$, $a \leqslant s \leqslant a+n-i-k$, and $D^k(i, ..., i+k-1)(a+n-i-k+1) > 0$, for $1 \leqslant i \leqslant r_k$, $1 \leqslant k \leqslant n$. (Note that this assertion is true for $1 \leqslant i \leqslant n-k+1$, $1 \leqslant k \leqslant n$ and so, it is trivially true for $1 \leqslant i \leqslant r_k$, $1 \leqslant k \leqslant n$.) By induction on k, we shall show that $D^k(i, ..., i+k-1)(s) > 0$, for $a+n-i-k+1 \leqslant s$ in $I^{n-i-k+2}$, $1 \leqslant i \leqslant r_k$, $1 \leqslant k \leqslant n$, A continuity argument will then be employed to construct an F-system with respect to $\{r_k\}$ of solutions of (1) on I^n.

Let $k=1$. Assume, for the sake of contradiction, that $D^1(i)(s) \leqslant 0$, for some $a+n-i<s$ in I^{n-i+1}, for some $1 \leqslant i \leqslant r_1 = l$. Assume without loss of generality that $D^1(i)(s-1) > 0$ and so $\Delta^{i-1}u_1$ has a generalized zero at s. Since $u_1(a) = \cdots = u_1(a+n-2) = 0$ by (10), $\Delta^{j-1}u_1(a+n-j-1) = 0$, if $1 \leqslant j \leqslant l$ and $l < n$, and $\Delta^{j-1}u_1(a+n-j-1) = 0$, if $1 \leqslant j \leqslant l-1$ and $l = n$. By repeated applications of Proposition 3.1, it follows $\Delta^{l-1}u_1$ has a generalized zero in $\{a+n-l, ..., s+i-l\}$.

There are two cases to consider.

(a) Assume that $m_l = 1$. By (10), u_1 has $n-1$ consecutive generalized zeros at $\{a, ..., a+n-2\}$. By Proposition 3.2, u_1 has an $m_1, ..., m_{l-1}$ right distribution of generalized zeros at $\{a, ..., a+n-l\}$. Since $\Delta^{l-1}u_1$ has a generalized zero in $\{a+n-l, ..., s+i-l\}$, it follows that u_1 has an

$m_1, ..., m_l$ right distribution of generalized zeros on $\{a, ..., s+i-l\}$. This contradicts (i).

(b) Assume that $m_l > 1$. Arguing as in (a), it follows that u_1 has an $m_1, ..., m_{l-1}$, $m_l - 1$ right distribution of generalized zeros at $\{a, ..., a+n-l-1\}$. Since $\Delta^{l-1}u_1$ has a generalized zero in $\{a+n-l, ..., s+i-l\}$, u_1 has an $m_1, ..., m_l$ right distribution of generalized zeros on $\{a, ..., s+i-l\}$ which, again, contradicts (i).

Thus, assertion (ii) holds for $k=1$.

Now, let $k>1$ and assume $D^\alpha(i, ..., i+\alpha-1)(s)>0$, $a+n-i-\alpha+1 \leqslant s$ in $I^{n-i-\alpha+2}$, $1 \leqslant i \leqslant r_\alpha$, $1 \leqslant \alpha < k$. Again, for the purpose of contradiction, assume that for some $1 \leqslant i \leqslant r_k$, and some $a+n-i-k+1 < s$ in $I^{n-i-k+2}$, that $D^k(i, ..., i+k-1)(s) \leqslant 0$. Assume without loss of generality that $D^k(i, ..., i+k-1)(s-1)>0$. Let $u = c_1u_1 + \cdots + c_{k-1}u_{k-1} + u_k$ where the constants $c_1, ..., c_{k-1}$ are chosen such that $\Delta^{i-1}u(s) = \cdots = \Delta^{i-1}u(s+k-2)=0$. From the induction hypothesis, the coefficients $c_1, ..., c_{k-1}$ are uniquely determined.

We now show that $\Delta^{i-1}u$ has k consecutive generalized zeros at $\{s, ..., s+k-1\}$. Note that by properties of determinants and elementary row operations, $D^k(i, ..., i+k-1)(s) = \Delta^{i-1}u(s+k-1)\, D^{k-1}(i, ..., i+k-2)(s)$. Thus, if $D^k(i, ..., i+k-1)(s)=0$, then $\Delta^{i-1}u(s+k-1)=0$ and $\Delta^{i-1}u$ has k consecutive generalized zeros at $\{s, ..., s+k-1\}$. If, on the other hand, $D^k(i, ..., i+k-1)(s)<0$, then

$$\begin{aligned} 0 &> D^k(i, ..., i+k-1)(s) \\ &= \Delta^{i-1}u(s+k-1)\, D^{k-1}(i, ..., i+k-2)(s) \end{aligned}$$

and

$$\begin{aligned} 0 &< D^k(i, ..., i+k-1)(s-1) \\ &= (-1)^{k-1}\, \Delta^{i-1}u(s-1)\, D^{k-1}(i, ..., i+k-2)(s). \end{aligned}$$

In particular, $(-1)^k\, \Delta^{i-1}u(s-1)\, \Delta^{i-1}u(s+k-1)>0$ and $\Delta^{i-1}u(s) = \cdots = \Delta^{i-1}u(s+k-2)=0$. Thus, $\Delta^{i-1}u$ has a generalized zero at $s+k-1$ and $\Delta^{i-1}u$ has k consecutive generalized zeros at $\{s, ..., s+k-1\}$.

Again, there are two cases to consider.

(c) First, assume that $r_k > r_{k+1}$, or that $k=n$. This corresponds to case (a) above. If $m_l=1$, then $r_1 = l > l-1 = r_2$. Since $r_k > r_{k+1}$, there is some $0 \leqslant j \leqslant l-1$ such that $k = m_l + \cdots + m_{l-j}$ and $r_k = l-j$. Note that $n-k = m_1 + \cdots + m_{l-j-1}$. $u_1, ..., u_k$ satisfy the partial set of initial conditions (10) and so $u(a) = \cdots = u(a+n-k-1)=0$; thus, u, where $u = c_1u_1 + \cdots + c_{k-1}u_{k-1} + u_k$ has been constructed above, has an $m_1, ..., m_{l-j-1}$ right distribution of generalized zeros at $\{a, ..., a+n-k-l+j+1\}$. Now, u has been constructed such that $\Delta^{i-1}u$ has k consecutive generalized zeros at

$s, ..., s+k-1$, where $a+n-i-k+1<s$ and $1 \leqslant i \leqslant r_k = l-j$. If $i=l-j$, $\Delta^{l-j-1}u$ has k consecutive generalized zeros at $s, ..., s+k-1$ and by Proposition 3.2, $\Delta^{l-j-1}u$ has an $m_{l-j}, ..., m_l$ right distribution of generalized zeros at $\{s, ..., s+k-l+i-1\}$. If $1 \leqslant i < l-j$, note that by the partial set of initial conditions (10), $\Delta^{h-1}u(a+n-k-h)=0$, $1 \leqslant h \leqslant l-j$. Thus, by repeated applications of Proposition 3.1, it follows that $\Delta^{l-j-1}u$ has k generalized zeros in $\{a+n-k-l+j+1, ..., s+k-l+j+i-1\}$. Hence, if $1 \leqslant i \leqslant r_k = l-j$, it follows from Proposition 3.2 that $\Delta^{l-j-1}u$ has an $m_{l-j}, ..., m_l$ right distribution of generalized zeros on $\{a+n-k-l+j+1, ..., s+k-l+i-1\}$. In particular, u has an $m_1, ..., m_l$ right distribution of generalized zeros on $\{a, ..., s+k-l+i-1\}$ and this contradicts (i).

(d) Now, assume $r_k = r_{k+1}$. This corresponds to case (b) above. Then, for some j, $1 \leqslant j \leqslant l-1$, $m_l + \cdots + m_{l-j+1} + 1 \leqslant k < m_l + \cdots + m_{l-j}$, or $1 \leqslant k < m_l$. Arguing as in case (c), it follows that u has an $m_1, ..., m_{l-j-1}$, $m_l + \cdots + m_{l-j} - k$ right distribution of generalized zeros at $\{a, ..., a+n-k-l+j\}$ and $\Delta^{l-j-1}u$ has a $k-m_l-\cdots-m_{l-j+1}, m_{l-j+1}, ..., m_l$ right distribution of generalized zeros on $\{a+n-k-l+j+1, ..., s+k-l+i-1\}$. This implies that u is an $m_1, ..., m_l$ right focal solution of (1) on I^n which contradicts (i). This completes the argument that $D^k(i, ..., i+k-1)(s)>0$, for $a+n-i-k+1 \leqslant s$ in $I^{n-i-k+2}$, $1 \leqslant i \leqslant r_k$, $1 \leqslant k \leqslant n$.

For $t \geqslant 0$, let $u_i^{(t)}(s)$, $1 \leqslant i \leqslant n$, be the system of solutions of (1) on I^n satisfying the initial conditions

$$\Delta^{i-1}u_k(a) = (-1)^{k-1}\, t^{n-i-k+1}/(n-i-k+1)!, \qquad 1 \leqslant i \leqslant n-k+1,$$

$$\Delta^{i-1}u_k(a) = 0, \qquad n-k+2 \leqslant i \leqslant n,\ 1 \leqslant k \leqslant n,$$

where $0^0 = 1$. Thus, u_i^0, $1 \leqslant i \leqslant n$, satisfies (10). Eloe [3] has shown that $D^k(i, ..., i+k-1)(s)>0$, $1 \leqslant i \leqslant n-k+1$, $1 \leqslant k \leqslant n$, $a \leqslant s \leqslant a+n-k$, where the system $u_i^{(t)}$, $1 \leqslant i \leqslant n$, is now the system employed in each determinant $D^k(i, ..., i+k-1)(s)$. Thus, $D^k(i, ..., i+k-1)(s)>0$, $1 \leqslant i \leqslant r_k$, $1 \leqslant k \leqslant n$, $a \leqslant s \leqslant a+n-k$. It now follows by continuity, as in [3], that for t sufficienty small, $u_i^{(t)}$, $1 \leqslant i \leqslant n$, is an F-system with respect to $\{r_k\}$ of solutions of (1) on I^n. This completes the proof of (i) implies (ii).

It is an immediate consequence of Lemma 2.3 that condition (ii) is equivalent to condition (iii).

We now address the equivalency of condition (ii) with condition (iv). It is clear that (iv) implies (ii), since by properties of determinants and elementary row operations, $D^k(i, ..., i+k-1)(s) = D^k(i, ..., i;\ s, s+1, ..., s+k-1)$. Thus, set $i_j = i$, $s_j = s+j-1$, $1 \leqslant j \leqslant k$, and (ii) follows from (iv) immediately.

To show that (ii) implies (iv), first define an ordering, which we call an antilexicographic ordering, on the set of indices satisfying

$$1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant l, \quad i_j \leqslant r_{k-j+1}, 1 \leqslant j \leqslant k. \tag{11_k}$$

For sets of indices $h_1, ..., h_k$ and $i_1, ..., i_k$ satisfying (11_k), we say

$$(h_1, ..., h_k) < (i_1, ..., i_k) \qquad \text{if and only if } h_\alpha < i_\alpha,$$

where $\alpha = \max\{\beta: h_\beta \neq i_\beta\}$.

The argument employs a double induction on k and the antilexicographic ordering. For $k = 1$, there is nothing to prove. Hence, assume $1 < k \leqslant n$ and assume $D^\alpha(i_1, ..., i_\alpha;\ s_1, ..., s_\alpha) > 0$, for all sets of indices $i_1, ..., i_\alpha$ satisfying (11_α) and all sets of points satisfying $a \leqslant s_j < s_{j+1}$ in $I^{n-i_{j+1}+1}$, if $i_j = i_{j+1}$, and $a \leqslant s_j \leqslant s_{j+1}$ in $I^{n-i_{j+1}+1}$, if $i_j < i_{j+1}$, $1 \leqslant j \leqslant \alpha - 1$, $1 \leqslant \alpha < k$.

For indices $i_1 = \cdots = i_k = 1$ and points $a \leqslant s_1 < \cdots < s_k$ in I^n, it follows from the Krein–Gantmacher criterion [6, Theorem 1, p. 283] that $D^k(1, ..., 1;\ s_1, ..., s_k) > 0$; see, Hartman [7, Theorem 5.1(g)]. In addition to the inductive assumption on k, assume that $(1, ..., 1) < (i_1, ..., i_k)$ and assume statement (iv) holds for all $(h_1, ..., h_k) < (i_1, ..., i_k)$.

Consider $D^k(i_1, ..., i_k;\ s_1, ..., s_k)$ where the i_q's and s_p's satisfy the conditions of (iv). Let $j = \max\{1 \leqslant \beta \leqslant k: i_\beta > i_{\beta-1}\}$ or set $j = 1$, if $i_1 = \cdots = i_k$. Then $i_j = \cdots = i_k$. There are two cases to consider depending on whether $s_{j-1} < s_j$ or $s_{j-1} = s_j$.

For the case $s_{j-1} < s_j$, it follows that $s_{j-1} < s_j < s_{j+1} < \cdots < s_k$, since $i_j = \cdots = i_k$. It follows from Lemma 2.1, with $b = (\Delta^{i_j-2}u_1(s_j), ..., \Delta^{i_j-2}u_k(s_j))$, that

$$\begin{aligned}
&D^{k-1}(i_2, ..., i_{j-1}, i_j - 1, i_j, ..., i_{k-1}; s_2, ..., s_{j-1}, s_j, s_j, ..., s_{k-1}) \\
&\quad \times D^k(i_1, ..., i_k; s_1, ..., s_k) \\
&\quad = D^{k-1}(i_1, ..., i_{k-1}; s_1, ..., s_{k-1}) \\
&\quad\quad \times D^k(i_2, ..., i_{j-1}, i_j - 1, i_j, ..., i_k; \\
&\quad\quad\quad s_2, ..., s_{j-1}, s_j, s_j, ..., s_k) \\
&\quad\quad + D^{k-1}(i_2, ..., i_k; s_2, ..., s_k) \\
&\quad\quad \times D^k(i_1, ..., i_{j-1}, i_j - 1, i_j, ..., i_{k-1}; \\
&\quad\quad\quad s_1, ..., s_{j-1}, s_j, s_j, ..., s_{k-1}).
\end{aligned}$$

By induction on k, each determinant D^{k-1} of order $k-1$ in the above expansion is positive. Moreover, $i_j = \cdots = i_k$ and so, $(i_1, ..., i_{j-1},\ i_j - 1,$

$i_j, ..., i_{k-1}) < (i_1, ..., i_k)$. Hence $D^k(i_1, ..., i_{j-1}, i_j - 1, i_j, ..., i_{k-1}; s_1, ..., s_{j-1}, s_j, s_j, ..., s_k) > 0$ by induction on the antilexicographic ordering. Finally, $s_j < s_{j+1}$ and so, $D^k(i_2, ..., i_{j-1}, i_j - 1, i_j, ..., i_k; s_2, ..., s_{j-1}, s_j, s_j, ..., s_k) = D^k(i_2, ..., i_{j-1}, i_j - 1, i_j - 1, i_{j+1}, ..., i_k; s_2, ..., s_{j-1}, s_j, s_j + 1, s_{j+1}, ..., s_k) > 0$ by induction on the antilexicographic ordering. Thus, $D^k(i_1, ..., i_k; s_1, ..., s_k) > 0$.

For the other case, $s_{j-1} = s_j$, again note that $s_{j-1} = s_j < s_{j+1} < \cdots < s_k$. There are two subcases to consider depending on whether $i_j = i_{j-1} + 1$ or $i_j > i_{j-1} + 1$.

If $i_j = i_{j-1} + 1$, then by properties of determinants and elementary row operations

$$\begin{aligned} D^k(i_1, ..., i_{j-1}, i_j, i_{j+1}, ..., i_k; s_1, ..., s_{j-1}, s_j, s_{j+1}, ..., s_k) \\ = D^k(i_1, ..., i_{j-1}, i_j - 1, i_{j+1}, ..., i_k; \\ s_1, ..., s_{j-1}, s_j + 1, s_{j+1}, ..., s_k). \end{aligned}$$

By the induction hypothesis on the antilexicographic ordering, the right-hand side of this equation is positive. Thus, $D^k(i_1, ..., i_k; s_1, ..., s_k) > 0$.

If $i_j > i_{j-1} + 1$, employ Lemma 2.1 as in the case $s_{j-1} < s_j$, with $b = (\Delta^{i_j - 2} u_1(s_j), ..., \Delta^{i_j - 2} u_k(s_j))$. It again follows that $D^k(i_1, ..., i_k; s_1, ..., s_k) > 0$ and the proof of (ii) implies (iv) is complete.

We now verify the last assertion that condition (iv) implies condition (i). Let $u_1, ..., u_n$ be an F-system with respect to $\{r_k\}$ of solutions of (1) on I^n; thus, the system of solutions $u_1, ..., u_n$ satisfies the positivity conditions of (iv). We show that there are no $m_1, ..., m_l$ right focal solutions of (1) on I^n. The proof relies on Proposition 3.3. Note here that each nontrivial solution u of (1) on I^n has the form $u = c_k(c_1 u_1 + \cdots + c_{k-1} u_{k-1} + u_k)$ for $c_1, ..., c_k \in \mathbf{R}$, $c_k \neq 0$, for some $1 \leqslant k \leqslant n$.

For $k = 1$, $u_1(s) = D^1(1; s) > 0$ on I^n; thus, u_1 is not an $m_1, ..., m_l$ right focal solution of (1) on I^n.

Let $k > 1$ and assume $1 < k \leqslant m_1$ or there is some α, $2 \leqslant \alpha \leqslant l$, such that $m_1 + \cdots + m_{\alpha-1} + 1 \leqslant k \leqslant m_1 + \cdots + m_\alpha$. Let $u = c_1 u_1 + \cdots + c_{k-1} u_{k-1} + u_k$ and assume that u has an $m_1, ..., m_{\alpha-1}, k - (m_1 + \cdots + m_{\alpha-1})$ right distribution of generalized zeros on I^n. Apply Proposition 3.3 and select $\sigma_1, ..., \sigma_{k-1}, \sigma_k$ such that

$$(-1)^{k-j+1} \Delta^{i-1} u(\sigma_j) \geqslant 0,$$

for each pair of indices $1 \leqslant i \leqslant \alpha$ and $1 \leqslant j \leqslant k$ satisfying $m_1 + \cdots + m_{i-1} + 1 \leqslant j \leqslant m_1 + \cdots + m_i$, and such that $a \leqslant \sigma_1 < \cdots < \sigma_{m_1} \leqslant \sigma_{m_1+1} < \cdots < \sigma_{m_1+m_2} \leqslant \cdots \leqslant \sigma_{m_1 + \cdots + m_{\alpha-1}+1} < \cdots < \sigma_k$.

Before we proceed, we introduce further notation. Let $\tau = (i_1, ..., i_k) \in \mathbf{R}^k$ such that $1 = i_1 = \cdots = i_{m_1}$, $2 = i_{m_1+1} = \cdots = i_{m_1+m_2}$, ..., $\alpha = i_{m_1 + \cdots + m_{\alpha-1}+1}$

$=\cdots=i_k$. For each $1 \leqslant j \leqslant k$, let $\tau(j) \in \mathbf{R}^{k-1}$ be obtained from τ by deleting the jth component. Let $\sigma=(\sigma_1, ..., \sigma_k) \in \mathbf{R}^k$ and for each $1 \leqslant j \leqslant k$, let $\sigma(j) \in \mathbf{R}^{k-1}$ be obtained from σ by deleting the jth component.

Substituting $u=u_k$ in the kth column of $D^k(\tau; \sigma)$, we obtain

$$D^k(\tau; \sigma)=\Delta^{\alpha-1}u(\sigma_k)\, D^{k-1}(\tau(k); \sigma(k)) + \sum_{j=1}^{k-1} (-1)^{k-j} \Delta^{i-1}u(\sigma_j)\, D^{k-1}(\tau(j); \sigma(j)).$$

By condition (iv), $0<D^k(\tau; \sigma)$; thus, by Proposition 3.3,

$$0 \leqslant \sum_{j=1}^{k-1} (-1)^{k-j+1} \Delta^{i-1}u(\sigma_j)\, D^{k-1}(\tau(j); \sigma(j)) < \Delta^{\alpha-1}u(\sigma_k)\, D^{k-1}(\tau(k); \sigma(k)).$$

In particular, $\Delta^{\alpha-1}u(\sigma_k)>0$. But this contradicts that $(-1)\,\Delta^{\alpha-1}u(\sigma_k) \geqslant 0$ by Proposition 3.3. Hence, u does not have an $m_1, ..., m_{\alpha-1}$, $k-(m_1+\cdots+m_{\alpha-1})$ right distribution of generalized zeros on I^n. In particular, u is not an $m_1, ..., m_l$ right focal solution of (1) on I^n. Thus, (1) is $m_1, ..., m_l$ right disfocal on I^n.

Remarks. (i) Consistent with the concept of the $m_1, ..., m_l$ right disfocality of (1) being between the disconjugacy of (1) and the right disfocality of (1), the F-system, (D-system) given in Theorem 3.4(ii) (Theorem 3.4(iii)) is between a Fekete system (Descartes system), necessary and sufficient for the disconjugacy of (1), and a D-Fekete system (D-Descartes system), necessary and sufficient for the right disfocality of (1).

(ii) There is an error in the proof of one of the lemmas in [3, Lemma 2.4]. However, the proof of (ii) implies (iv) in Theorem 3.4, given here, can be employed to obtain several of the lemmas in [3, Lemmas 2.4, 2.5, and 2.6].

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